NON – NEWTONIAN HYDROMECHANICS OF DISPERSE SYSTEMS

PMM Vol. 32, No. 3, 1968, pp. 381-392

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(Received January 3, 1968)

We construct a statistical theory of a fluid – suspended particles system, under the assumption that random, localized fluctuations of the phase velocities and concentrations are imposed on the averaged flow. We obtain the criterion of the violation of the homogeneity of the flow in such a system and the expressions for the transfer coefficients connected with the fluctuating motion as well as the corresponding dynamic equations.

When we attempt the description of the motion of a disperse system making use of the Navier – Stokes equations for a homogeneous fluid with certain effective viscosity coefficients, we run into expected difficulties related to the fact that these coefficients are dependent on the intensity of these localized phase fluctuations imposed on the average motion of the flow. Therefore neither a satisfactory mechanical model of disperse systems, nor the formulation of the corresponding dynamic equations can be realized without a detailed analysis of these random motions.

Particles suspended in a viscous fluid form a complex, nonconservative system which possesses both, the properties of a dense gas with potential interaction, and the distinctive features of an interacting Brownian system of particles. The required statistical theory can be constructed by two different methods. The first of these methods is based on the assumption that the particles of the system exhibit small amplitude motions causing porosity fluctuations, while groups of particles exhibit large amplitude fluctuations caused by the interaction of small amplitude fluctuations with the supporting flow [1 and 2]. The second method is based on the study of the kinetic equation for the suspended particles, under the assumption that their motion resulting from their interaction with the dispersing medium can be simulated by a random process with independent parameters [3 and 4]. We have shown in [2] that the latter assumption is false, since the process of variation of random velocities of the particles is not Markovian. The assumption that the particles move about in a bunch, is also largely conventional since the spatial fluctuations are of the order of the mean distance between the particles of the system. This dimension, however, is not small enough to justify the adoption of the hypothesis of statistical independence of individual particles which, like the well known hypothesis of the random molecular motion in the kinetic theory of gases, plays the decisive part in the second of two methods mentioned above.

The statistical theory of disperse systems considered below is based, as in [2], on the correlation theory of random processes, but without separating the pulsating motion into small and large amplitude components.

1. Stochastic equations. We consider a monodisperse system of particles of radius a and density d_2, suspended in a medium of density d_1 and viscosity μ_0. We shall use the two-velocity model, in which the phases are considered to be mutually permeable, interacting, continuous media [5]. Equations of conservation of mass of the phases under the assumption of incompressibility of both the particles and the dispersing medium, have the form

$$-\frac{\partial \rho}{\partial t}+\nabla[(1-\rho)\mathbf{v}]=0, \qquad \frac{\partial \rho}{\partial t}+\nabla(\rho\mathbf{w})=0, \qquad \nabla=\frac{\partial}{\partial \mathbf{r}} \tag{1.1}$$

Where ρ denotes the mean volume concentration of the dispersed phase connected with the mean porosity ϵ of the system by the relation $\rho = 1 - \epsilon$. Equations of conservation of impulse of the phases can be written in the form

$$d_1\frac{\partial[(1-\rho)\mathbf{v}]}{\partial t}=-\nabla\Pi^{(1)}+d_1(1-\rho)\mathbf{g}-\mathbf{F}$$

$$d_2\frac{\partial(\rho\mathbf{w})}{\partial t}=-\nabla\Pi^{(2)}+d_2\rho\mathbf{g}+\mathbf{F} \tag{1.2}$$

$$\Pi^{(l)}=\Pi_0{}^{(l)}+\Pi^{(l)'} \quad (l=1,\ 2), \qquad \Pi_0{}^{(1)}=p\mathbf{E}+d_1(1-\rho)(\mathbf{v}*\mathbf{v})$$

$$\Pi_0{}^{(2)}=d_2\rho(\mathbf{w}*\mathbf{w})$$

Here p is the pressure in the fluid phase, F is the intensity of the interaction between the phases per unit volume of the disperse system, g is the gravity vector and $\mathbf{E}$ is a unit tensor. Phase impulse flux density tensors $\Pi^{(l)}$ are represented as sums of the 'regular' and unregular' terms the latter of which describes the impulse transfer caused by phase fluctuations and by the viscous coupling in the dispersing medium. Symbol $\mathbf{a} * \mathbf{b}$ denotes a dyadic product of the vectors $\mathbf{a}$ and $\mathbf{b}$.

Hydrodynamic parameters entering (1.1) and (1.2) are obtained by averaging over the volumes containing a very large number of particles. E.g. if the velocity of the j-th particle is $\mathbf{w}^{(j)}$, its specific volume σ_j and the mean velocity of the fluid phase within this volume is $\mathbf{v}^{(j)}$, then we have the following expressions for the mean parameters:

$$\rho=\lim_{N\to\infty}\sum_{j=1}^{N}\frac{\sigma_0}{\sigma_j}, \qquad \mathbf{v}=\lim_{N\to\infty}\frac{1}{(1-\rho)V}\sum_{j=1}^{N}(\sigma_j-\sigma_0)\mathbf{v}^{(j)}, \qquad \mathbf{w}=\lim_{N\to\infty}\frac{1}{\rho V}\sum_{j=1}^{N}\sigma_0\mathbf{w}^{(j)} \tag{1.3}$$

where σ_0 denotes the volume of a single particle.

Neglecting the Basse force, we can write for $\mathbf{F}$ the following expression:*

$$\mathbf{F}=-\rho\nabla p+d_1\rho\xi\frac{\partial \mathbf{u}}{\partial t}+\beta\rho K\mathbf{u}, \quad \mathbf{u}=\mathbf{v}-\mathbf{w}, \quad \beta=\frac{9}{2}\frac{\mu_0}{a^2} \tag{1.4}$$

Here $\xi=\xi(\rho)$ is the coefficient of the apparent mass and $K = K(\rho)$ is a function taking into account the deviation of the viscous force from the Stokes force when the flow around the particles is constrained. We consider ξ and K to be certain experimentally determinable functions of ρ.

The vector fields $\mathbf{v}(t, \mathbf{r})$, $\mathbf{w}(t, \mathbf{r})$, $\rho(t, r)$ and $p(t, \mathbf{r})$ satisfying (1.1), are assumed known. Corresponding magnitudes governed by real volumes containing finite, even if large number of particles, differ from these mean values by small fluctuations $\mathbf{v}'$, $\mathbf{w}'$ ρ' and p'. Equations for the fluctuations can be obtained, after the Eqs. (1.1) and (1.2) are linearized, and due regard should be given to the fact that the 'hydrodynamic' expressions for $\Pi^{(l)'}$ $(l = 1, 2)$, given later in Section 4 can only be linearized when the amplitudes of the perturbations exceed those of the fluctuations by a large amount. When the dynamic Eqs. (1.2) are linearized with respect to these fluctuations we must, obviously, take into account the force of fluctuation $f^{(l)}$ exerted on the phases by the surrounding medium in a unit volume, in their explicit form. Assuming that the forces acting on the phases are proportional to the relative areas occupied by these phases on the surface of the unit volume we find, that $\mathbf{f}^{(1)} = (1 - \rho)\mathbf{f}$, and $\mathbf{f}^{(2)} = \rho\mathbf{f}$, where $\mathbf{f}$ is a random vector whose mean value is zero.

* A different expression for $\mathbf{F}$ was proposed by the author in [6], but it was later found erroneous due to the disregard of inertial forces which appear during the use of the accelerating coordinate systems.

Using a coordinate system in which $\mathbf{w} = 0$ and $\mathbf{v} = \mathbf{u}$ we obtain from (1.1) the following Eqs. for fluctuations

$$-\left(\frac{\partial}{\partial t} + \mathbf{u}\nabla\right)\rho' + (1-\rho)\nabla\mathbf{v}' - (\nabla\rho)\mathbf{v}' - (\nabla\mathbf{v})\rho' = 0$$

$$\frac{\partial\rho'}{\partial t} + \rho\nabla\mathbf{w}' + (\nabla\rho)\mathbf{w}' + (\nabla\mathbf{w})\rho' = 0 \tag{1.5}$$

Taking (1.1) and (1.4) into account we obtain, after linearizing (1.2)

(1.6)

$$d_1(1-\rho)\left[\left(\frac{\partial}{\partial t} + \mathbf{u}\nabla\right)\mathbf{v}' + (\mathbf{v}'\nabla)\mathbf{v}\right] + \rho\left(d_1\xi\frac{\partial}{\partial t} + \beta K\right)(\mathbf{v}' - \mathbf{w}') = (1-\rho)\mathbf{f}' + \mathbf{A}_1\rho'$$

$$d_2\rho\left[\frac{\partial\mathbf{w}'}{\partial t} + (\mathbf{w}'\nabla)\mathbf{w}\right] - \rho\left(d_1\xi\frac{\partial}{\partial t} + \beta K\right)(\mathbf{v}' - \mathbf{w}') = \rho\mathbf{f}' + \frac{\rho}{1-\rho}\mathbf{A}_2\rho'$$

$$\mathbf{f}' = \mathbf{f} - \nabla p' \qquad \mathbf{A}_1 = d_1\left(\frac{\partial}{\partial t} + \mathbf{u}\nabla\right)\mathbf{v} - d_1\frac{d(\rho\xi)}{d\rho}\frac{\partial\mathbf{u}}{\partial t} + \nabla p - d_1\mathbf{g} - \beta\frac{d(\rho K)}{d\rho}\mathbf{u}$$

$$\frac{\rho}{1-\rho}\mathbf{A}_2 = -d_2\frac{\partial\mathbf{w}}{\partial t} + d_1\frac{d(\rho\xi)}{d\rho}\frac{\partial\mathbf{u}}{\partial t} - \nabla p + d_2\mathbf{g} + \beta\frac{d(\rho K)}{d\rho}\mathbf{u}$$

We note that unlike the method given in [2], the present method utilizes, as an unperturbed solution, not some fictitious motion corresponding to a well-ordered array serving as a model of a disperse syste, but a real averaged motion. In this respect, our method resembles that of Enskog in the kinetic theory.

Following [2] we shall represent all fluctuations by stochastic Fourier-Stielties integrals with random integrating function Z containing independent increments [7] to obtain, from (1.5) and (1.6), differential equations for these functions

$$[i(1-\rho)\mathbf{k} - \nabla\rho]\,d\mathbf{Z}_v = [i(\omega + \mathbf{uk}) + \nabla\mathbf{v}]\,dZ_\rho$$

$$(i\rho\mathbf{k} + \nabla\rho)\,d\mathbf{Z}_w = -(i\omega + \nabla\mathbf{w})\,dZ_\rho$$

$$d_1(1-\rho)[i(\omega + \mathbf{uk})\,d\mathbf{Z}_v + (d\mathbf{Z}_v\nabla)\mathbf{v}] + \rho(id_1\xi\omega + \beta K)(d\mathbf{Z}_v - d\mathbf{Z}_w) = (1-\rho)\,d\mathbf{Z}_f + \mathbf{A}_1 dZ_\rho \tag{1.7}$$

$$d_2\rho[i\omega d\mathbf{Z}_w + (d\mathbf{Z}_w\nabla)\mathbf{w}] - \rho(id_1\xi\omega + \beta K)(d\mathbf{Z}_v - d\mathbf{Z}_w) = \rho d\mathbf{Z}_f + \rho(1-\rho)^{-1}\mathbf{A}_2 dZ_\rho$$

where ω is the frequency and $\mathbf{k}$ is the wave vector.

2. Expressions for random processes. Below we assume that vector fluctuations in a flow can be obtained by superimposing the isotropic and anisotropic waves, i.e. for any vector φ' we have

$$d\mathbf{Z}_\varphi = \mathbf{k}\,dZ_\varphi^\circ + d\mathbf{Z}_\varphi', \qquad \varphi = \mathbf{v},\ \mathbf{w},\ \mathbf{f} \tag{2.1}$$

Velocity fluctuations in the liquid phase represent a secondary effect in the sense that their appearance is wholly governed by the requirement that the mass is conserved during the random motion of the particles. It is reasonable to assume that the change in the number of particles in a fixed stationary volume causes only the isotropic fluctuations $\mathbf{v}$, while the appearance of anisotropic motions of the dispersing medium is caused only by the convective transfer of the porosity fluctuations, i.e. $d\mathbf{Z}_v'$ becomes zero as $\mathbf{u} \to 0$.

We note that an essentially analogous assumption was used in [1, 2 and 4]. Then, the first Eq. of (1.7) yields

$$dZ_v^\circ = U^\circ dZ_\rho, \qquad d\mathbf{Z}_v' = \mathbf{U}\,dZ_\rho$$

$$U^\circ = \frac{i\omega + \nabla\mathbf{v}}{i(1-\rho)k^2 - \mathbf{k}\nabla\rho}, \qquad U_m = \frac{iu_m k_m}{i(1-\rho)k_m - \nabla_m\rho}, \qquad \nabla_m = \frac{\partial}{\partial x_m} \tag{2.2}$$

We notice that when $\rho \approx$ const, the vector $\mathbf{U}$ coincides with $\mathbf{u}$. Inserting (2.1) and (2.2) into the third Eq. of (1.7), we obtain

$$dZ_f^\circ = -\rho(1-\rho)^{-1}(i\omega d_1\xi + \beta K)\, dZ_w^\circ + (1-\rho)^{-1}B_1 dZ_\rho$$

$$d\mathbf{Z}_f' = -\rho(1-\rho)^{-1}(i\omega d_1\xi + \beta K)\, d\mathbf{Z}_w' + (1-\rho)^{-1}\,\mathbf{C}_1 dZ_\rho$$

$$B_1 = U^\circ\,[i\omega d_1(1-\rho+\rho\xi) + id_1(1-\rho)\,\mathbf{uk} + \beta\rho K]$$

$$\mathbf{C}_1 = -\mathbf{A}_1 + U^\circ d_1(1-\rho)(\mathbf{k}\nabla)\,\mathbf{v} + \mathbf{U}\,[i\omega d_1(1-\rho+\rho\xi) + id_1(1-\rho)\,\mathbf{uk} + \beta\rho K] + d_1(1-\rho)(\mathbf{U}\nabla)\,\mathbf{v}$$

which, employed in the fourth Eq. of (1.7) yield, after some manipulations, the following expressions for dZ_w° and $d\mathbf{Z}_w'$

$$dZ_w^\circ = B\,[i\omega(d_2(1-\rho) + d_1\xi) + \beta K]^{-1} dZ_\rho, \qquad B = B_1 + B_2$$

$$d\mathbf{Z}_w' = \mathbf{H}^{-1}\mathbf{C}\,dZ_\rho, \qquad \mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2, \qquad B_2 = (1-\rho)(i\omega d_1\xi + \beta K)\,U^\circ$$

$$\mathbf{C}_2 = \mathbf{A}_2 + (1-\rho)(i\omega d_1\xi + \beta K)\,\mathbf{U} - d_2(1-\rho)B\,[i\omega(d_2(1-\rho) + d_1\xi) + \beta K]^{-1}(\mathbf{k}\nabla)\,\mathbf{w}$$

$$\mathbf{H} = [i\omega(d_2(1-\rho) + d_1\xi) + \beta K]\,\mathbf{E} + d_2(1-\rho)\,\mathbf{W}, \qquad W_{ij} = \nabla_i w_j$$

Eqs. (2.1) to (2.3) define completely the required random processes as functions of the random quantity dZ_ρ and of the mean parameters.

If the spatial and temporal variations of the mean values of parameters are macroscopic, while the corresponding fluctuation amplitudes of the disperse system are microscopic, we can apply the concepts of continuous media to the disperse system and assume, that

$$\frac{\partial \ln\varphi}{\partial t} \sim \delta\omega, \qquad \frac{\partial \ln\varphi}{\partial r} \sim \delta k, \qquad \delta \ll 1$$

where ϕ denotes any mean parameter of the flow (except p). A set of equations of consecutive approximations in terms of a small δ can be easily obtained and it can easily be seen that the first approximation in which the terms of the lowest order in δ are retained, is analogous to the hydrodynamic approximation leading to the Navier – Stokes equations, the second approximation of the kinetic theory is analogous to the Barnett approximation, etc. These approximations lead to considerable simplification of all equations.

In general, neglecting the derivatives of the mean parameters, we obtain

$$dZ_v^\circ = \frac{\omega_i^2}{(1-\rho)k^2}\,dZ_\rho, \quad d\mathbf{Z}_v' = \frac{\mathbf{u}}{1-\rho}\,dZ_\rho$$

$$dZ_w^\circ = \frac{B\,dZ_\rho}{i\omega(d_2(1-\rho) + d_1\xi) + \beta K}, \quad d\mathbf{Z}_w' = \frac{\mathbf{C}\,dZ_\rho}{i\omega(d_2(1-\rho) + d_1\xi) + \beta K} \tag{2.4}$$

$$\mathbf{C} = \frac{1}{\rho}\left[-\nabla p + (d_1\rho + d_2(1-\rho))\,\mathbf{g} + \beta\frac{d(\rho K)}{d\rho}\,\mathbf{u}\right] + \frac{\mathbf{u}}{1-\rho}\,[id_1(1-\rho)(\omega + \mathbf{uk}) + i\omega d_1\xi_j' + \beta K] = \frac{\mathbf{R}}{\rho} + \frac{\mathbf{u}S}{1-\rho}, \quad B = \frac{\omega S}{(1-\rho)k^2}$$

To compute the various correlations, we must first obtain the spectral density $\Psi_{\rho\rho}(\omega, \mathbf{k})$ of the random process ρ'. An expression for the spectral density of this process (only in the wave space) describing the simultaneous correlations, was obtained in [2] under the assumption of statistical independence of the position of various particles in space. It is given by

$$\Phi_{\rho\rho}(k) = \frac{3\sigma_0}{8\pi^3}\frac{\rho(\rho_* - \rho)}{\rho_*}\frac{\sin kb - kb\cos kb}{(kb)^3}, \qquad b = b(N) = a\left(\frac{N}{\rho}\right)^{1/3} \quad (2.5)$$

where ρ_* is the concentration of the system when it is densely packed, while b denotes the radius of a volume containing, on the average, N particles. The averaging process during the determination of fluctuations is performed over this volume. The choice of N, i.e. trunkation of the short-wave part of the fluctuation spectrum is dictated, in the present theory, by the minimum size of the fluctuations at which the linearization in Section 1 can be performed. The most detailed 'fine grain' description of the structure of our disperse corresponds to the choice of the specific volume of a single particle as an elementary volume (see the discussion in [2]).

Introduction in (2.5) of the last multiplier dependent on k, corresponds to the process of smoothing of the short wave irregularities of the spectrum, which was proposed by Massignon in [8]. Alternatively, we can use a well known procedure of Debye, in which the spectral density is

$$\Phi_{\rho\rho}(k) = \frac{3\sigma_0}{8\pi^3}\frac{\rho(\rho_* - \rho)}{\rho_*}Y(k_\infty - k), \qquad k_\infty = \left(\frac{3\pi}{2}\right)^{1/3}\frac{1}{b} \quad (2.6)$$

where $Y(x)$ is a Heaviside's function. Such an expression was used in [1].

We note that in the presence of a fluid phase in the system, the positions of the neighboring particles cannot, in general, be assumed statistically independent. This however becomes unessential when the fluctuations are taken over sufficiently large values ($N \gg 1$).

The dynamics of the degeneracy of the porosity fluctuations and, consequently, the spectral density $\Psi_{\rho\rho}(\omega, \mathbf{k})$ depend, in the end, on the amount of deviation of the system from equilibrium, and this is the basic cause of the difficulties encountered in the determination of $\Psi_{\rho\rho}(\omega, \mathbf{k})$. To overcome these difficulties, we separated in [2] all fluctuations into their micro- and macro-components, assuming at the same time that the variations in porosity fluctuations with time represent a diffusion process governed only by the small amplitude motions. In fact, the rates of degeneracy are governed by the local fluctuations of all mean parameters, and ought to be described by the same Eqs. (1.5) and (1.6) as the solution of a certain problem with given initial data.

Inserting (2.2) and (2.3) into the second Eq. of (1.7), we obtain (2.7)

$$M(i\omega)\, dZ_\rho = 0, \quad M(i\omega) = (i\omega + \nabla w)\,[i\omega\,(d_2(1-\rho) + d_1\xi) + \beta K] + (i\rho\mathbf{k} + \nabla\rho)\,\{\mathbf{k}\,B(i\omega) + [i\omega\,(d_2(1-\rho) + d_1\xi) + \beta K]\,\mathbf{H}^{-1}\mathbf{C}(i\omega)$$

Eq. $M(\lambda) = 0$ represents a characteristic equation of the linear system (1.5) and (1.6), and its roots define the mode of decay of the fluctuations in the problem with given initial data.

In fact, the degeneration of fluctuations caused by the regular factors allowed for in the equations, is compensated by random accumulation of fluctuations, which, naturally, are not accounted for by this equations. An analogous phenomenon is well known in the kinetic theory: the Boltzmann equation for example, or the transfer equation ensuing from it, describe an approximation to the equilibrium state with maximum entropy, neglecting the – so called – background noise resulting from the local deviations from the molecular randomness [9]. In our system, such a background noise may appear e.g. as a result of small aplitude perturbations within the specific volumes, and these perturbations cannot, in principle, be described by means of the regularized Eqs. (1.5) and (1.6). Therefore, when the accumulation of fluctuations is taken into account, then the following stochastic equation corresponds to (2.7)

$$M_0\left(\frac{\partial}{\partial t}\right)dY_\rho = \Delta(t, \mathbf{k}), \qquad dY_\rho(t, \mathbf{k}) = \int e^{i\omega t}\, dZ_\rho(\omega, \mathbf{k}) \quad (2.8)$$

Here $M_0(\lambda)$ is a polynomial obtained by multiplying $M(\lambda)$ by the common denominator of the quantities contained within the curly brackets in (2.7), $\Delta(t, \mathbf{k})$ is a random function of t depending also on $\mathbf{k}$ as a parameter and of constant spectral density, while the integration in (2.8) is performed over the whole frequency range.

Thus (2.8) finally yields the following Formula

$$\Psi_{\rho\rho}(\omega, \mathbf{k}) = \frac{\Phi_{\rho\rho}(k)}{|M_0(i\omega)|^2}\left(\int \frac{d\omega}{|M_0(i\omega)|^2}\right)^{-1} \tag{2.9}$$

Writing an approximation corresponding to (2.4), we obtain the following expression for $M_0(i\omega)$

$$M_0(i\omega) = -(\omega^2 + 2a_1\omega + a_2^2) + i(2b_1\omega + b_2^2) \tag{2.10}$$

$$2a_1 = \frac{d_1}{d_0}\left(2 + \frac{\xi}{1-\rho}\right)\rho u\mathbf{k}, \qquad 2b_1 = \frac{\beta K}{(1-\rho)d_0}, \qquad a_2^2 = \frac{d_1}{d_0}\rho(u\mathbf{k})^2$$

$$b_2^2 = \frac{1}{d_0}\left(\mathbf{R} + \frac{\rho\beta K}{1-\rho}\mathbf{u}\right)\mathbf{k}, \qquad d_0 = d_2(1-\rho) + d_1\left(\rho + \frac{\xi}{1-\rho}\right)$$

$$|M_0(i\omega)|^2 = (\omega^2 + 2a_1\omega + a_2^2)^2 + (2b_1\omega + b_2^2)^2$$

By virtue of (2.4), relations (2.9) and (2.10) also define the spectral densities of other random processes considered here.

3. Structure of the steady, zero-gradient flows and critical fluctuations. Correlation functions can be computed using known spectral densities, from the following Formula [7]:

$$r_{\alpha\gamma}(t, \mathbf{r}; \tau, \boldsymbol{\xi}) = \langle \alpha(t+\tau, \mathbf{r}+\boldsymbol{\xi})\gamma^*(t, \mathbf{r})\rangle = \iint e^{i(\omega\tau + \mathbf{k}\boldsymbol{\xi})}\Psi^*_{\alpha\gamma}(\omega, \mathbf{k})\,d\omega\,d\mathbf{k} \tag{3.1}$$

where α and γ denote arbitrary random processes.

Let us first consider the fluctuations taking place within large volumes ($N \gg 1$) so that $k \leqslant b^{-1}$ can be considered as a small parameter during the integration over the frequencies. In this case, we have

$$\frac{a_1}{b_1} \sim \frac{a_2}{b_1} \sim \left(\frac{b_2}{b_1}\right)^2 \ll 1 \tag{3.2}$$

and the polynomial $|M_0(i\omega)|^2$ from (2.10) can be approximately represented by

$$|M_0(i\omega)|^2 \approx (\omega^2 + 4b_1^2)\left[\left(\omega + \frac{b_2^2}{2b_1}\right)^2 + \frac{1}{4}\left(\frac{b_2^4}{4b_1^2} + \frac{a_2^2}{b_1} - \frac{a_1b_2^2}{b_1^2}\right)^2\right]$$

Let us consider the steady, zero-gradient flows, and orientate the coordinate axes so, that $\mathbf{g} = (-g, 0, 0)$.

Then Eqs. (1.2) yield the following relations for these flows

$$\nabla_x p = -[d_1(1-\rho) + d_2\rho]\,g, \qquad \beta K u = -(1-\rho)(d_2 - d_1)\,\mathbf{g} \tag{3.3}$$

From (2.10) we obtain, after some manipulations, the following expression for $|M_0(i\omega)|^2$

$$|M_0(i\omega)|^2 \approx (\omega^2 + c_1^2)[(\omega + c_2k_1)^2 + c_3^2k_1^4]$$

$$c_1 = \frac{\beta K}{(1-\rho)d_0}, \qquad c_2 = \rho(1-\rho)\left(\frac{2}{1-\rho} + \frac{d\ln K}{d\rho}\right)u$$

$$c_3 = \frac{1-\rho}{\beta K}\left|d_0c_2^2 + \rho d_1 u\left[u - \left(2 + \frac{\xi}{1-\rho}\right)c_2\right]\right|$$

Using the approximate equation $\omega^2 + c_1^2 \approx c_1^2$, we obtain, on integrating (2.9),

$$\Psi_{\rho\rho}(\omega,\ \mathbf{k}) = \frac{c_1^2 c_3 k_1^2 \Phi_{\rho\rho}(k)}{\pi(\omega^2 + c_1^2)\left[(\omega + c_2 k_1)^2 + c_3^2 k_1^4\right]} \tag{3.4}$$

Integrating the spectral density tensor $\mathbf{v}'$ of the process obtained from (2.4) and using either (2.5) or (2.6) together with (3.4), we obtain the approximate expression for the correlation tensor $\langle v_i' v_j' \rangle$

$$\theta_{ij}^{(1)} = r_{vi,\,j}(t,\ \mathbf{r};\ 0,\ 0) = \langle v_i' v_j' \rangle =$$

$$= \frac{1}{N}\left(\frac{\rho}{1-\rho}\right)^2 \frac{\rho_* - \rho}{\rho_*}\left[\Phi_1(\rho,\ \varkappa)\,\delta_{ij} + (1 + 2\Phi_1(\rho,\ \varkappa))\,\delta_{i1}\delta_{j1}\right] u^2$$

$$\Phi_1(\rho,\ \varkappa) = \frac{\rho}{15}\left|\rho(1-\rho)^2\left(\frac{2}{1-\rho} + \frac{d\ln K}{d\rho}\right)^2 + \frac{\varkappa}{1-\rho+\varkappa(\rho+(1-\rho)^{-1}\xi)} \times \right. \tag{3.5}$$

$$\left. \times\left[1 - \left(2 + \frac{\xi}{1-\rho}\right)\rho(1-\rho)\left(\frac{2}{1-\rho} + \frac{d\ln K}{d\rho}\right)\right]\right|, \qquad \varkappa = \frac{d_1}{d_2}$$

Similarly, for $\langle w_i' w_j' \rangle$ we obtain

$$\theta_{ij}^{(2)} = r_{wi,\,j}(t,\ \mathbf{r};\ 0,\ 0) = \langle w_i' w_j' \rangle =$$

$$= \frac{1}{N}\left(\frac{\rho}{1-\rho}\right)^2 \frac{\rho_* - \rho}{\rho_*}\left[\Phi_2(\rho,\ \varkappa)\,\delta_{ij} + (\Phi_0(\rho) + 2\Phi_2(\rho,\ \varkappa))\,\delta_{i1}\delta_{j1}\right] u^2$$

$$\Phi_0(\rho) = (1-\rho)^2\left(\frac{2}{1-\rho} + \frac{d\ln K}{d\rho}\right)^2, \qquad \Phi_2(\rho,\ \varkappa) = \left\{(1-\rho)^2 + \right. \tag{3.6}$$

$$\left. + \varkappa\left[\rho(1-\rho) + \xi\right] + \varkappa^2\frac{(1-\rho+\xi)^2}{1-\rho+\varkappa\xi}\right\}\frac{\Phi_1(\rho,\ \varkappa)}{(1-\rho)(2-\rho) + \varkappa\left[\rho(1-\rho) + 2\xi\right]}$$

Last two expressions show, that the intensity of the longitudinal fluctuations of both phases is much higher than that of the transverse fluctuations. At large N, the computed quadratic fluctuations are proportional to N^{-1}. This implies that the positions of the single particles can indeed be assumed to be statistically independent and Formulas (2.5) and (2.6) used. Moreover, this dependence of the quadratic fluctuations on N, justifies the use of (1.3) in the determination of the mean values of the parameters. For example, at large N, we have

$$\mathbf{Q}_1 = (1-\rho)\,\mathbf{v} - \langle \rho'\mathbf{v}' \rangle \approx (1-\rho)\,\mathbf{v}, \qquad \mathbf{Q}_2 = \rho\mathbf{w} + \langle \rho'\mathbf{w}' \rangle \approx \rho\mathbf{w}$$

for the phase flux.

Let us now decrease N, i.e. turn our attention to small scale local properties of the disperse system. Consider two limiting cases: 1° – when the inequalities (3.2) hold when k increases up to the value $k_m = \max\{k\} \sim a^{-1}\rho^{1/3}$, which corresponds to the most detailed description of the fluctuations and 2° – when the inequalities (3.2) cease to hold at some $k = k_0$ and in which the inequality is reversed, i.e. $b_2 \gg b_1, a_1, a_2$ when $k = k_m$. From (2.10) we easily see, that the first case is typical for the relatively fine, lightweight suspensions in liquids of high density and viscosity, while the other is typical for suspensions of large, heavy particles in a gas. In the first case additional terms appear in the expressions for the tensors $\theta^{(l)}$ $(l = 1, 2)$. These terms are proportional to the higher negative powers of N and this indicates the presence of negative correlation constraints in the system. In the second case, we have the following approximate relations for $k \gg k_0$ replacing (3.4)

$$|M_0(i\omega)|^2 \approx \omega^4 + b_2^4, \qquad \Psi_{\rho\rho}(\omega,\ \mathbf{k}) \approx \frac{\sqrt{2}b_2^3}{\pi}\frac{\Phi_{\rho\rho}(k)}{\omega^4 + b_2^4} \tag{3.7}$$

Thus for $\theta_{ji}^{(1)}$, say, we have, using for simplicity the expression for $\Phi_{\rho\rho}(k)$ given by (2.6),

$$\theta_{ij}^{(1)} = \left(\frac{\rho}{1-\rho}\right)^2 \frac{\rho_* - \rho}{\rho_*}\left[\frac{1}{N}u^2\delta_{i1}\delta_{j1} + \left(\frac{3\pi}{2N}\right)^{2/3}\Psi_0(\rho, \varkappa)\, ag\,(\delta_{ij} + \delta_{i1}\delta_{j1})\right] \quad (3.8)$$

$$\Psi_0(\rho, \varkappa) = \frac{1}{8\pi}\frac{\rho^{2/3}(1-\rho)(1-\varkappa)}{1-\rho+\varkappa[\rho+(1-\rho)^{-1}\xi]}\left(\frac{2}{1-\rho} + \frac{d\ln K}{d\rho}\right)$$

Expressions for other correlation functions exhibit, in this case, similarly to (3.8), the terms proportional to $N^{-2/3}$, and this indicates the presence of positive correlation constraints in the system. Thus in both cases, single particles and the fluid elements are found, within their specific volumes, to be statistically dependent, and Formulas (2.5) and (2.6) invalid. In this connection, the system under consideration resembles classical systems with long range interaction (ionised gas with Coulombic interaction, ordinary gas in the critical region, etc.). We find, that the characteristics of random phase motions referred to relatively small elementary volumes (e.g. to the specific volume) can be computed, within the framework of the theory based on the relations (2.5) and (2.6), only approximatively, by extrapolating expressions of the type (3.5) and (3.6) to small N.

Here we must question the validity of the method of statistical analysis of disperse systems based on the investigation of the kinetic equation for one-particle (unitary) distribution function, when a statistical connection exists between the neighboring particles. Indeed we find, that the random forces existing between the particles and the fluid phase which enter this equation, are represented by the functionals, which are not only the functionals of the unitary, but also of the binary and higher, multi-particle distribution functions. Somewhat surprisingly, this is the case for gas suspensions of the coarse particles, which were studied earlier by means of this method [4].

Clearly, the systems with negative correlation constraits have small amplitude fluctuations ('homogeneous' systems). Conversely, systems with positive correlation constraints ('inhomogeneous' systems) exhibit large amplitude fluctuations which may, in principle, result in the appearance of group movement of the particles, formation of large scale aggregates and formation of bubbles filled with dispersing medium and containing, practically, no particles. It is possible, that it is precisely those critical fluctuations that are responsible for the observed transition from the homogeneous to the nonhomogeneous mode of pseudo fluidization [10] in the manner similar to that occurring in gases, where the critical fluctuations cause the appearance of macroscopic volumes of the condensed phase and other associated phenomena such as that of the critical opalescence etc. [11 and 12]. Comparing b_2 with $2b_1$ and using the relations (2.10) and (3.3) we obtain, for $k_1 \sim \rho^{1/3}a^{-1}$, the following condition of the appearance of the nonhomogeneous mode:

$$(AL(\rho, \varkappa))^{1/2} \gtrless 1, \qquad A = \frac{8a^3g}{\nu_0^2}\left(\frac{1}{\varkappa} - 1\right), \qquad \nu_0 = \frac{\mu_0}{d_1}$$

$$L(\rho, \varkappa) = \frac{1}{160}\left(\frac{1-\rho}{\varkappa} + \rho + \frac{\xi}{1-\rho}\right)\frac{\rho^{4/3}(1-\rho)^3}{K^2(\rho)}\left(\frac{2}{1-\rho} + \frac{d\ln K}{d\rho}\right) \quad (3.9)$$

where A is the Archimedes criterion. The decisive part played by the value of A in establishing the homogeneity or inhomogeneity of the pseudo-fluidized state, was deduced earlier from empirical considerations, by several authors. In [13] for example, the appearance of bubbles in the disperse layer is related to a parameter, which has the same meaning as A. In general we find, that the condition (3.9) shows a good agreement with many experiments on the violation of the homogeneous state and transition to the inhomogeneous state (see the survey in [10]. If (3.9) holds, then the same concepts make it possible to estimate the linear 'equilibrium' dimension l of the inhomogeneities in the system. We have

$$l \sim AL(\rho, \varkappa)\, a \quad (3.10)$$

Investigation of the critical fluctuations in an inhomogeneous state can, apparently, be conducted with the help of a certain function, differing from the δ-function, which defines the correlation coefficients between the porosities in nonoverlapping volumes. This will alter the form of Expressions (2.5) and (2.6) for $\Phi_{\rho\rho}(k)$. Such an approach was successfully employed by Ornshtein and Tsernike when analyzing the fluctuations of gases in the critical region [11].

It should be stressed, that Formulas (3.5) to (3.10) refer to the steady, zero-gradient flows. Corresponding relations for the general case can be obtained in a similar manner, although purely computational problems may present serious difficulties. It seems that computations performed separately for various types of disperse flows, constitute a more rewarding approach.

4. Dynamic equations. Final formulation of the mechanical model demands explicit expressions for the impulse flux density tensor in (1.2) and for the transport coefficients in both phases. Rigorous expressions for these coefficients can only be obtained from the sequence of equations for partial distribution functions, and this sequence has not yet been derived. We shall use the phenomenological approach. In particular, we shall assume that when the gradients are small, then the tensors $\Pi^{(l)\prime}$ from (1.2) can be written in the form corresponding to a homogeneous, anisotropic fluid. We then have

$$\begin{aligned}\Pi^{(1)\prime} &= \mathbf{P}^{(1)} - (\eta^{(1)} + \mu\mathbf{E})\,[2\Gamma^{(1)} - {}^2/_3(\nabla\mathbf{v})\,\mathbf{E}] - \zeta^{(1)}(\nabla\mathbf{v}) \\ \Pi^{(2)\prime} &= \mathbf{P}^{(2)} - \eta^{(2)}\,[2\Gamma^{(2)} - {}^2/_3(\nabla\mathbf{w})\,\mathbf{E}] - \zeta^{(2)}(\nabla\mathbf{w})\end{aligned} \tag{4.1}$$

Here $\mathbf{P}^{(l)}$, $\eta^{(l)}$, and $\zeta^{(l)}$ $(l = 1,2)$ denote, respectively, the pressure (flux density of the reversible impulse transport) tensor and the shear and volume viscosity tensors of the fluid and dispersed phase dependent on their pulsating motions, while $\mu = \mu(\rho)$ is the effective viscosity of the fluid filtering through an ordered lattice of stationary particles. Viscosity μ is proportional to μ_0, increases with increasing ρ and for small ρ it satisfies the following approximate expression $\mu \approx \mu_0 (1 + {}^5/_2\,\rho)$ (see e.g. [14]).

The following approximate expressions for the fluid phase are obtained by analogy with the turbulence in a homogeneous fluid [7]

$$\mathbf{P}^{(1)} \approx (1-\rho)\,d_1\Theta^{(1)}, \qquad \eta^{(1)} \approx (1-\rho)\,d_1\mathbf{D}^{(1)}, \qquad \zeta^{(1)} \approx 0 \tag{4.2}$$

$$\mathbf{D}^{(1)} \approx \frac{1}{2}\int_0^\infty \{\mathbf{R}_v(t,\ \mathbf{r};\ \tau,\ \tau\mathbf{u}) + \mathbf{R}_v^*(t,\ \mathbf{r};\ \tau,\ \tau\mathbf{u})\}\,d\tau$$

$$\Theta^{(1)} = \mathbf{R}_v(t,\ \mathbf{r};\ 0,\ 0) = \lim_{N\to 1}\theta^{(1)} = \lim_{N\to 1}\mathbf{r}_v(t,\ \mathbf{r};\ 0,\ 0)$$

The tensors $\Theta^{(1)}$, and $\mathbf{R}_v$ appearing here, refer to the fluid within the specific volume of a single particle and are obtained from the magnitudes defined for a volume element containing N particles ($N \gg 1$) (see Section 3). Molecular diffusion and molar mixing of fluid due to its motion along curved paths between the particles in the lattice, are not taken into account in (4.2).

We can distinguish between two limit modes of flow [1]. They will depend on the relationship between the mean time T of traverse of the mean free path λ by a particle, and the time T_0 of relaxation of this particle to the conditions existing in the supporting flow. In the first, 'pseudo-turbulent' mode, we have $T \sim \lambda \cdot \langle |w^{(l)}|^2 \rangle^{-1/2} \gg T_0 = nm\,(\beta K)^{-1}$, $m = \sigma_0 d_2$, $n = \rho / \sigma_0$, and the collisions between the particles have no appreciable influence on the local mass and impulse transport processes. In the

second, 'pseudo-gaseous' state, $T \ll T_0$ and the collisions are mainly responsible for the transport processes in the disperse phase. When $\rho \ll 1$, $\lambda \approx \lambda_0$ where λ_0 denotes the classical free path; when $\rho \leqslant \rho_*$, the magnitude λ may represent the difference in the radii of specific volumes in the state under consideration and in the state of close packing. We have

$$\lambda \approx a\,(\rho\rho_*)^{-1/3}\,(\rho_*^{1/3} - \rho^{1/3}) \sim \rho_* - \rho$$

It follows from Section 3 that the magnitude $\langle |w^{(j)}|^2 \rangle$ is proportional to $\rho_* - \rho$, and T_0 tends to the value different from zero as $\rho \to \rho_*$. Therefore the 'pseudo-gaseous' state may accomodate an arbitrary disperse system, provided that the latter is of sufficient concentration. As $\rho \to 0$, $T \to \infty$ and the 'pseudo-turbulent' state may materialize in any, sufficiently diluted system.

An obvious analogy between the transport in the system of suspended particles and the transport due to the turbulent fluctuations in a homogeneous medium, can also be noted in the 'pseudo-turbulent' state.

The following approximate expressions are analogous to those given for the fluid phase [7]

$$\mathbf{P}^{(2)} \approx \rho d_2 \Theta^{(2)}, \quad \eta^{(2)} \approx \rho d_2 \mathbf{D}^{(2)}, \quad \zeta^{(2)} \approx 0$$

$$\mathbf{D}^{(2)} \approx \frac{1}{2}\int_0 \{\mathbf{R}_w(t,\ \mathbf{r};\ \tau,\ 0) + \mathbf{R}_w^*(t,\ \mathbf{r};\ \tau,\ 0)\}\,d\tau \tag{4.3}$$

In the 'pseudo-gaseous' flow we have a clearly defined analogy between the suspended particles and a gas composed of rigid spheres. This analogy was utilised in [1 and 4]. In this case the tensor $\mathbf{P}^{(2)}$ and the transport coefficients can be approximately estimated from the known results of the kinetic theory of dense gases by multiplying the quantities given by the elementary theory, by some functions of ρ. Using, as in [1], the results of the Enskog's theory of dense gases, we obtain the following approximate relations

$$\mathbf{P}^{(2)} \approx \rho d_2 (1 + Y(\rho))\,\Theta^{(2)}, \qquad \eta^{(2)} \approx 4\rho\eta^\circ\,[Y^{-1}(\rho) + 0.8 + 0.76Y(\rho)]$$

$$\zeta^{(2)} \approx 4\rho\eta^\circ Y(\rho), \qquad \mathbf{D}^{(2)} \approx 4\rho\mathbf{D}^\circ Y^{-1}(\rho), \qquad Y(\rho) = 4\rho\chi(\rho) \tag{4.4}$$

$$\eta^\circ = \rho d_2 \mathbf{D}^\circ, \qquad D_{ij}^\circ = 2\lambda_0\,(\Theta_{ij}^{(2)})^{1/2}, \qquad \lambda_0 = (4\sqrt{2}\pi a^2 n)^{-1}, \qquad n = \rho\sigma_0^{-1}$$

Function $\chi(\rho)$ shows how the frequency of the binary collisions (we note that in the system composed of spheres with δ-type interaction, all collisions are binary) changes with concentration. As we know, increase in the collision frequency with increasing ρ is caused by diminishing of the free volume within which the centers of spheres are free to move, while the screening action of the neighboring particles on each particle leads to the decrease in frequency. When $\rho \ll 1$, then the expression obtained by Enskog for $\chi(\rho)$ and utilised in the theory of disperse systems, holds [4]

$$\chi(\rho) \approx (1 - {}^{11}/_2\ \rho)\,(1 - 8\rho)^{-1},\ \rho \ll 1$$

In his earlier work [1], the author used the expression for $\chi(\rho)$ obtained from the approximate 'geometrical' theory of dense gases, which gave satisfactory results for $\rho > 0.10$ to 0.15.

$$\chi(\rho) \approx \frac{1}{4\rho}\,\frac{(\rho/\rho_*)^{1/3}}{1 - (\rho/\rho_*)^{1/3}} = \frac{1}{4\rho^{2/3}(\rho_*^{1/3} - \rho^{1/3})} \sim \frac{1}{\rho_* - \rho}$$

The appearance of the function $\chi(\rho)$ in the expression for $\mathbf{P}^{(2)}$ in (4.4) is connected with the fact that an impulse is transmitted instantaneously within the material of the particle.

Thus, in general, the rheological characteristics of disperse systems depend strongly on the type of flow and are usually represented by very complicated functions of mean parameters of the flow and of their derivatives with respect to the spatial and time coordinates. When $\rho \to 0$, all magnitudes in (4.2) and (4.3) decrease as ρ^{α} where $\alpha > 0$. When $\rho \to \rho_*$, then the components of $\mathbf{P}^{(2)}$ tend to the values differing from both, zero and infinity since $\Theta^{(2)} \sim \rho_* - \rho$, while $\chi(\rho) \sim (\rho_* - \rho)^{-1}$, the components of $\mathbf{D}^{(2)}$ tend to zero as $(\rho_* - \rho)^{3/2}$, while $\eta^{(2)}$ and $\xi^{(2)}$ tend to infinity as $(\rho_* - \rho)^{-1/2}$. Various components of last two tensors considered as functions of ρ can, and usually have a maximum and a minimum when $\rho \sim \rho_*$. Such a dependence of the viscosity on ρ has been obtained experimentally for dense, disperse systems (see e.g. [15]).

The problem of formulating a unique equation of conservation of impulse for the total amount of dispersed phase, is interesting. We can infer from (1.2), that this is indeed possible in two cases: (1) – for the particles suspended in a gas, when we can neglect the inertia and the gas fluctuations and express $\mathbf{F}$ and $\mathbf{v}$ from the first Eq. of (1.2) in terms of ρ, ∇p and $\mathbf{w}$, subsequently inserting them into (1.1) and the second Eq. of (1.2), and (2) – for the suspensions of almost uniform density when we can assume that $\mathbf{u} \approx 0$ and sum Eqs. (1.2). In both cases we have three equations; one of the conservation of impulse of the dispersed phase and two of the conservation of mass of both phases, for three unknowns $\mathbf{w}$, ρ and p. We note, that the boundary conditions assume in these cases a slightly different form; in particular, the pressure gradient of the fluid phase appears in these conditions.

From this we see, that the disperse systems are, essentially, non-Newtonian. Their rheological properties are represented by very complicated functions of mean parameters of the flow, and this leads to the appearance of additional nonlinearities in the dynamic Eqs. (1.2). In addition, rheological characteristics of the same system may differ appreciably from one case to the other, depending on the type of motion effected by the system.

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Translated by L.K.